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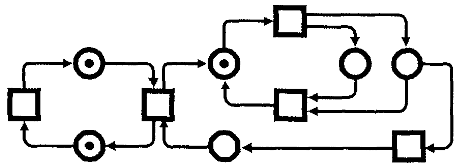

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Formal Grounding of Toudic Method

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Abstract

Paper gives a formal grounding of know heuristic Toudic method for the solution of linear diophantine homogeneous systems of equations in nonnegative integer numbers. In particular nonnegative solutions are required for calculation of the invariants for Petri nets. At first we construct the solution for one equation. Then the solution is expanded to whole system of equations. To generate all solutions via the Toudic's basis a linear combination was extended by special operation of reduction in common measure of vectors' components.

Keywords: Diophantine equation; Linear system; Nonnegative solution; Petri net; Invariant

1. Introduction

The problem of the solution of linear diophantine homogeneous systems of equations arises in the Petri net theory [3] during the process of net invariants calculation and also in other fields of computer science. Invariants are powerful tools for investigation the major properties of the Petri net. Its allow to determine boundness, safeness, necessary conditions of liveness and deadlock-free. These properties are significant for real objects analysis [3] especially communication protocols, enterprise systems, computer hardware and software.

It is known a heuristic method proposed by Toudic [1]. This method allows obtain nonnegative integer solutions by the means of matrix transformations. Method was introduced without a formal grounding. At start we have identity matrix that at finish contain the basis solutions. But the completeness of the basis was not proved. Now do not looking at asymptotic exponential complexity this method is widely used in enterprise Petri net analysis tools. This paper gives a formal grounding of Toudic method.

It is wide known in the mathematics a lot of methods to calculate solution of the system of linear equation in rational numbers. A variety of those include theoretical constructions based on calculation of matrix determinants, classical Gauss method [4] and also special numerical methods introduced for the minimization of computational error.

If we consider an integer matrix of the system and calculate an integer solutions then a special methods are required. This group of methods is based on unimodular transformations of matrix [6] to obtain Smith normal form. It is known special polynomial algorithms [2] used solutions in the classes field of deductions modulo of prime numbers to construct a target general integer solution of the system.

In the Petri net theory the solutions of the state equation [3] are the transition firing count vectors and have to be a nonnegative. Homogeneous systems are used for obtaining of the net invariants that are a powerful tool for the structural properties of Petri net investigation [3]. It is the special problem to construct a basis of nonnegative integer solutions of a system. In this paper this problem is solved for homogeneous system. To generate all solutions from Toudic's basis we introduce a special operation of vector reduction in common measure of components.

2. Solution of one equation

We start from the construction of solution for one equation. So we have

$$\bar{a}\cdot\bar{x}-\bar{b}\cdot\bar{y}=0 \qquad (1)$$

where $\bar{a},\bar{x},\bar{b},\bar{y}$ are nonnegative integer vectors; dimension of $\bar{a},\bar{x}$ is *m*, dimension of $\bar{b},\bar{y}$ is *n*; $\bar{a},\bar{b}$ are known coefficients; $\bar{x},\bar{y}$ are unknown variables.

Theorem 1. An arbitrary solution of (1) may be represented as $\frac{\bar{z}\cdot G}{k}$, where *k* is a common measure of $\bar{z}\cdot G$ vector components; $\bar{z}=(z_1^1,z_2^1,...,z_n^1,z_1^2,z_2^2,...,z_n^2,...,z_1^m,z_2^m,...,z_n^m)$ - an nonnegative integer vector,

$$G=\begin{vmatrix} b_1 & 0 & .. & 0 & a_1 & 0 & .. & 0 \\ b_2 & 0 & .. & 0 & 0 & a_1 & .. & 0 \\ : & : & : & : & : & : & : & : \\ b_n & 0 & .. & 0 & 0 & 0 & .. & a_1 \\ 0 & b_1 & .. & 0 & a_2 & 0 & .. & 0 \\ 0 & b_2 & : & 0 & 0 & a_2 & .. & 0 \\ : & : & : & : & : & : & : & ; \\ 0 & b_n & .. & 0 & 0 & 0 & .. & a_2 \\ & . & & & & . & & \\ & . & & & & . & & \\ & . & & & & . & & \\ 0 & 0 & .. & b_1 & a_m & 0 & .. & 0 \\ 0 & 0 & .. & b_2 & 0 & a_m & .. & 0 \\ : & : & : & : & : & : & : & 0 \\ 0 & 0 & .. & b_n & 0 & 0 & .. & a_m \end{vmatrix}$$

In other words, rows of matrix G are basis solutions for each pair (b_i,a_j). A formal description of matrix *G* may be provided as following. Let us denote $\bar{g}^l$ *l*-th row of matrix *G*. Then $\bar{g}^l, l=\overline{1,m\times n}$ has such nonzero components as

$$g_{l,((l-1)\ div\ n)+1}=b_{((l-1)\ \mathrm{mod}\ n)+1},$$
$$g_{l,m+((l-1)\ \mathrm{mod}\ n)+1}=a_{((l-1)\ div\ n)+1}$$

Proof. Thus, we have $(\bar{x},\bar{y})=\dfrac{\bar{z}\cdot G}{k}$. Let us $(\bar{c},\bar{d})$ is an arbitrary solution of (1). We have to prove that exists $k,\bar{z}$ so that

$$k\cdot(\bar{c},\bar{d})=\bar{z}\cdot G \tag{2}$$

We write more detailed representation of (2)

$$\begin{cases}\sum_i b_i\cdot z_i^j-c_j\cdot k=0, & j=\overline{1,m}\\ \sum_j a_j\cdot z_i^j-d_i\cdot k=0, & i=\overline{1,n}\end{cases} \tag{3}$$

The proof will be executed by a constructive approach. We suggest the concrete solution of the system (3)

$$\begin{cases}k=\bar{a}\cdot\bar{c} \quad (or \quad k=\bar{b}\cdot\bar{d})\\ z_i^j=c_j\cdot d_i, \quad j=\overline{1,m}, i=\overline{1,n}\end{cases} \tag{4}$$

Now we demonstrate that (4) is the solution of (3):

$$\sum_i b_i\cdot c_j\cdot d_i-c_j\cdot\bar{a}\cdot\bar{c}=c_j\cdot\bar{b}\cdot\bar{d}-c_j\cdot\bar{a}\cdot\bar{c}=c_j\cdot(\bar{b}\cdot\bar{d}-\bar{a}\cdot\bar{c})=0, \quad j=\overline{1,m}$$

$$\sum_j a_j\cdot c_j\cdot d_i-d_i\cdot\bar{a}\cdot\bar{c}=d_i\cdot\bar{a}\cdot\bar{c}-d_i\cdot\bar{a}\cdot\bar{c}=0, \quad i=\overline{1,n}.$$

So (4) is the solution of (3). □

It is convenient to introduce a special operation for an arbitrary integer vector with nonnegative components. This operation $<\bar{x}>$ calculates the results of vector $\bar{x}$ reduction in common measure of components so if $\bar{y}=<\bar{x}>$ then a natural k such as $k\cdot\bar{y}=\bar{x}$ exists.

Hence, solutions of (1) may be represented in the form

$$\bar{x}=<\bar{z}\cdot G>.$$

Theorem 2. Basis presented in Theorem 1 is a minimal.

Proof. Let us choose in the matrix *G* an arbitrary basis solution

$$\bar{g}_i^j=(0,\ldots,b_i,\ldots,0,0,\ldots,a_j,\ldots,0) \tag{5}$$

Now we demonstrate that solution (5) can not be obtained from the residuary basis solutions (rows of matrix *G*) by means of such operations as sum of vectors multiplied by a constants and by division on integer.

Really, any solution that has nonnegative component in the position *j* has also nonnegative component in any position *l*, $l\neq j$. And then we can not obtain zero in this position with the operations of addition, multiplication and division on nonnegative number. □

So in this section was obtained a formal method for the solution of one equation. A basis consists of $m\times n$ vectors represented by the matrix *G*.

3. Solution of a system of equations

Now we consider a system

$$\bar{x}\cdot A=0, \tag{6}$$

where *A* is a given integer $m\times n$ matrix, $\bar{x}$ is an unknown nonnegative integer vector.

We introduce the transformations of matrixes. We shall be obtaining a matrix *D* from the matrix *A* with two elementary transformations:

1) Write line *y* into line *v*:

$\bar{l}^v \leftarrow \bar{l}^y$

2) Write the sum of lines *y* and *z* multiplied by a constants into line *v*:

$\bar{l}^v \leftarrow c_y \cdot \bar{l}^y + c_z \cdot \bar{l}^z$

In such case the matrix *D* can be obtained from matrix *A* by means of multiplying from the left on matrix of transformation *R*. Matrix *R* has such nonzero components:

I. For every transformation 1):

$r_{v,y} = 1$

II. For every transformation 2):

$r_{v,y} = c_y, \ \ r_{v,z} = c_z$

It may be easily proved if we consider the representation of *D*:

$$D = R \cdot A, \ \ d_{i,j} = \sum_k r_{i,k} \cdot a_{k,j} \ .$$

Our method of system solution is similar to a Gauss method. It is based on the solution of one equation and substitution of the obtained general solution into the residuary part of system. We process until all equations will be solved and we will obtain so a zero matrix.

At first consider a solution of one (for example first) equation of (6). So we have equation

$\bar{x} \cdot \bar{a}^1 = 0$

It may be easily constructed I^+, I^0, I^- such sets of indexes as

$I^+ = \{i \mid a_{i,1} > 0\}, \ \ I^0 = \{i \mid a_{i,1} = 0\}, \ \ I^- = \{i \mid a_{i,1} < 0\}$.

Further, according to Theorem 1 we construct a matrix of solutions with transformations I, II:

$$\bar{x} = \bar{z} \cdot \left| \begin{array}{c} (I^0)^I \\ \hline (I^+ \times I^-)^{II} \end{array} \right| = \left| \begin{array}{c} (I^0)^I \\ \hline G \end{array} \right| = \bar{z} \cdot R \, .$$

So

$\bar{x} = < \bar{z} \cdot R >$ or $k \cdot \bar{x} = \bar{z} \cdot R$ (7)

We multiply (6) by a natural number k and substitute (7) in (6)

$\bar{z} \cdot R \cdot A = 0$

or

$\bar{z} \cdot D = 0$, (8)

where $D = R \cdot A$. It has to be mentioned that solution of an equation adds new variables to the system.

Theorem 3. System (8), (7) is equivalent to the system (6).

Proof. Transformations above prove the necessary condition. It shall be proved the sufficient one. We replace in (8) $\bar{z} \cdot R$ according to (7) and obtain

$k \cdot \bar{x} \cdot A = 0$

So k is a natural, we divide above equation by k and obtain (6). □

Now we consider the process of the consecutive solution for equations of the system.

$\bar{x} = < \bar{z}^1 \cdot R^1 >$

$\bar{z}^1 \cdot R^1 \cdot A = 0$

Continuing in such a manner we obtain

$\bar{z}^n \cdot R^n \cdot R^{n-1} \cdots R^1 \cdot A = 0$

Let us denote

$\bar{z} := \bar{z}^n$, $R := R^n \cdot R^{n-1} \cdots R^1$

then

$\bar{z} \cdot R \cdot A = 0$, $R \cdot A = 0$,

since at each step we reduce a one equation of the system. So we have

$\bar{z} \cdot 0 = 0$.

Hence, an arbitrary vector $\bar{z}$ is the solution of the system. The final solution of the system (6) may by represented as

$\bar{x} = < \bar{z} \cdot R >$ or $k \cdot \bar{x} = \bar{z} \cdot R$. (11)

So according to Theorem 3 we used on each step the equivalent (reversible) transformations, we have proved the following theorem.

Theorem 4. Expression (11) represents all the nonnegative integer solutions of system (6).

Hence it's required find a matrix R to solve the system. Matrix R contains basis solutions. To generate an arbitrary solution a linear combination and reduction in common measure of vectors' components are used.

4. Description of the algorithm

Expression (11) may be written as

$$\bar{x} = < \bar{z} \cdot R \cdot E >,$$

where E is identity matrix. Then accounting the process of matrix R construction we obtain

$R = R^n \cdot R^{n-1} \cdots R^1 \cdot E$.

So to calculate matrix R it is required to repeat all the transformations of the source matrix A with the identity matrix E.

Hence, we come to Toudic's [1] description of the algorithm.

Algorithm:

Step 0. Let us $D := A$, $R := E$.

Step 1. If $D = 0$ then Stop. Matrix R is the result matrix of basis solutions of system.

Step 2. If all the columns of matrix D contain nonzero coefficients of the same sign then Stop. System is inconsistent and has only trivial solution.

Step 3. Choose an arbitrary column j of matrix D with a minimal value of product $|I^+| \times |I^-|$, where $I^+ = \{i \mid a_{i,j} > 0\}$, $I^0 = \{i \mid a_{i,j} = 0\}$, $I^- = \{i \mid a_{i,j} < 0\}$.

Step 4. Construct matrix D' in such a manner: copy to matrix D' rows I^0 of matrix D and then write additional rows for each combination of $(k,r), k \in I^+, r \in I^-$ created as $|a_{r,j}| \cdot \bar{l}^k + |a_{k,j}| \cdot \bar{l}^r$.

Step 5. Execute the same transformations with matrix R to construct matrix R'.

Step 6. Assign $D := D', R := R'$ and goto Step 1.

At Step 3 the column choice provides the minimal quantity of new solutions for corresponding equation. According to previous results rows of the matrixes D and R may be divided together in a common measure of components at any step of the algorithm.

Note that a peculiarity is in the manner of obtained basis R usage to generate all the solutions. We expand the linear combination with nonnegative integer coefficients with a new operation of reduction in common measure of vectors' components. So the operation of the division by natural number does not produce new zero or nonzero components then additional operation of the reduction does not influent on invariant supports' obtaining. Note that support [1,3] is a set of nonzero components of the solution.

5. Example consideration

It will be solved the following concrete system of equations to illustrate obtained results:

$$\begin{cases} 5\cdot x_1 + 5\cdot x_2 - 2\cdot x_3 - 2\cdot x_4 = 0 \\ 2\cdot x_1 - 5\cdot x_2 - 5\cdot x_3 - 5\cdot x_5 = 0 \\ 5\cdot x_1 + 2\cdot x_2 + 2\cdot x_3 - 2\cdot x_4 - 5\cdot x_5 = 0 \end{cases} \qquad (12)$$

In the matrix form (6) we have

$$\bar{x} = (x_1, x_2, x_3, x_4, x_5),\ A = \begin{pmatrix} 5 & 5 & -2 & -2 & 0 \\ 2 & -5 & -5 & 0 & -5 \\ 5 & 2 & 2 & -2 & -5 \end{pmatrix}^T .$$

We write the pair of matrixes (D, R):

$$\left|\begin{array}{ccc|ccccc} 5 & 2 & 5 & 1 & 0 & 0 & 0 & 0 \\ 5 & -5 & 2 & 0 & 1 & 0 & 0 & 0 \\ -2 & -5 & 2 & 0 & 0 & 1 & 0 & 0 \\ -2 & 0 & -2 & 0 & 0 & 0 & 1 & 0 \\ 0 & -5 & -5 & 0 & 0 & 0 & 0 & 1 \end{array}\right|$$

The minimal value of $|I^+| \times |I^-| = 3$ is obtained for the second column; it will be chosen as a column *j*. So, we have $j = 2$ and calculate

$$\left|\begin{array}{ccc|ccccc} -2 & 0 & -2 & 0 & 0 & 0 & 1 & 0 \\ 35 & 0 & 29 & 5 & 2 & 0 & 0 & 0 \\ 21 & 0 & 29 & 5 & 0 & 2 & 0 & 0 \\ 25 & 0 & 15 & 5 & 0 & 0 & 0 & 2 \end{array}\right|$$

Further the column $j = 1$ will be chosen

$$\left|\begin{array}{ccc|ccccc} 0 & 0 & -12 & 10 & 4 & 0 & 35 & 0 \\ 0 & 0 & 16 & 10 & 0 & 4 & 21 & 0 \\ 0 & 0 & -20 & 10 & 0 & 0 & 25 & 4 \end{array}\right|$$

After processing the column $j = 3$ we obtain

$$\begin{vmatrix} 0 & 0 & 0 & 280 & 64 & 48 & 203 & 0 \\ 0 & 0 & 0 & 360 & 0 & 80 & 820 & 64 \end{vmatrix}$$

And finally, reducing first and second rows by 4, we obtain basis solutions

$$R = \begin{pmatrix} 70 & 16 & 12 & 203 & 0 \\ 90 & 0 & 20 & 205 & 16 \end{pmatrix}.$$

It is easily may be calculated that vector

$$\bar{b} = \begin{pmatrix} 20 & 2 & 4 & 51 & 2 \end{pmatrix}$$

is the solution of the system (12) but it can not be obtained as a linear combination of basis solutions with nonnegative coefficients. With the operation of reduction it may be represented as

$$\bar{b} = (\bar{r}^1 + \bar{r}^2)/8.$$

Analogously we may calculate

$$(150,24,28,407,8) = (3 \cdot \bar{r}^1 + \bar{r}^2)/2,$$

$$(170,8,36,409,24) = (\bar{r}^1 + 3 \cdot \bar{r}^2)/2,$$

$$(130,4,28,307,20) = (4 \cdot \bar{r}^1 + 20 \cdot \bar{r}^2)/16.$$

6. Conclusions

In present work a formal grounding for known as heuristic and widely used in computer tools Toudic method of Petri net invariants calculation was given. It was shown that Toudic algorithm provides a set of basis solutions containing all invatiants' supports.

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