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Solving the fundamental equation of Petri net using the decomposition into functional subnets

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Abstract

The technique of solution of the fundamental equation of Petri net based on decomposition into functional subnets is proposed. Solutions of the fundamental equation of the entire Petri net are calculated out of solutions of the fundamental equations of functional subnets for dual Petri net. Acceleration of computations obtained is exponential with the respect to dimension of Petri net.

Keywords: *Petri net, Fundamental equation, Decomposition, Functional subnet*

1. INTRODUCTION

Matrix methods [1,2,14] are the most prospective for large-scale real-life systems' Petri net models analysis. The fundamental equation of Petri net constitutes a system of linear Diophantine equations [5]. Solutions of this system are interpreted as the firing count vectors for the allowed sequences of transitions and so have to be nonnegative integer numbers that stipulates the specifics of the task. Methods for these systems solution are represented in [3,4,6,9,13]. Unfortunately, all the known methods have an asymptotically exponential complexity that makes its application for real-life systems analysis difficult.

The goal of the present work is the construction of the compositional methods for solution of fundamental equation of Petri net allowing the considerable acceleration of computations. Really, models of complex systems are assembled out of models of its components usually. Moreover, in the cases the composition of model out of subnets is not given, we suggest to apply the methods of Petri net decomposition represented in [8] to partition of a given Petri net into set of its functional subnets [7]. Earlier the analogous technique was applied successfully for invariants calculation [10,15].

The acceleration of computations obtained is estimated with an exponential function. Since the dimension of subnets as a rule is essentially little than the dimension of entire net, the actual acceleration of computations may be extremely considerable that was confirmed by the results of this technique application to communication protocols analysis [11,12,16].

2. BASIC CONCEPTS

Petri net is a quadruple $N=(P,T,F,W)$, where $P=\{p\}$ is a finite set of nodes named places, $T=\{t\}$ is a finite set of nodes named transitions; flow relation $F \subseteq P\times T \bigcup T\times P$ defines a set of arcs connecting places and transitions, mapping $W: F \to \mathrm{N}$ defines a multiplicity of arcs; N denotes a set of natural numbers.

Marking of net is a mapping $\mu : P \to \mathrm{N}_0$, defining a distribution of dynamic elements named tokens over places; N_0 is a set of nonnegative integer numbers. *Marked Petri net* is a couple $M=(N,\mu_0)$ or a quintuple $M=(P,T,F,W,\mu_0)$, where μ_0 is initial marking.

A *fundamental equation* of Petri net [5] may be represented as follows

$$\bar{x} \cdot A = \Delta\bar{\mu} \tag{1}$$

, where $\Delta\bar{\mu} = \bar{\mu} - \bar{\mu}_0$, $\bar{x}$ is a firing count vector, A is a transposed incidence matrix or incidence matrix of dual Petri net [5]. Notice that each equation of this system corresponds to a transition of dual Petri net.

It is known [1,2,5] that the solvability of fundamental equation in nonnegative integer numbers is a necessary condition of the reachability of a given marking. Solutions of system (1) are used for the construction of the required firing sequences.

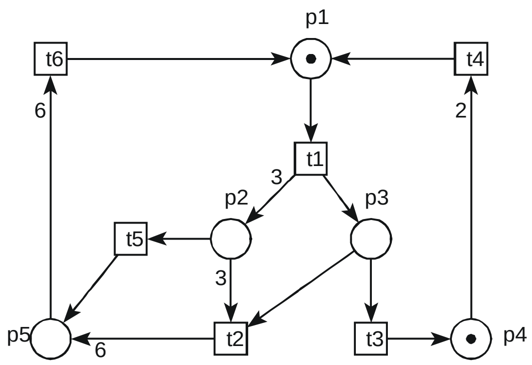

Fig. 1. Petri net N_1

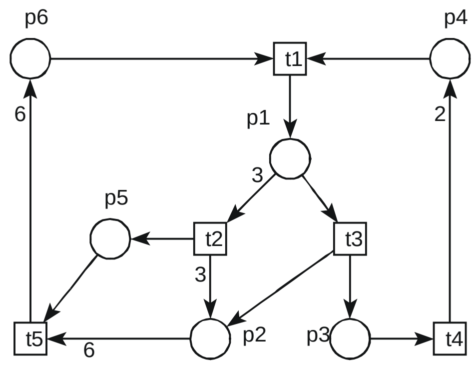

Fig. 2. Dual Petri net $\tilde{N}_1$

According to [3,4] we shall represent a general solution of homogeneous system as the linear combination of basis solutions with nonnegative integer coefficients. Notice that a basis consists of minimal in integer nonnegative lattice solutions of system. As distinct from classic theory of linear systems for representation of general nonnegative integer solution of nonhomogeneous system it is necessary to involve not one arbitrary but a set of minimal particular solutions.

3. FUNCTIONAL SUBNETS

Net with input and output places is Petri net the special subsets of places namely input and output are indicated in which.

Functional net is a triple $Z = (N, X, Y)$, where N is Petri net, $X \subseteq P$ is a set of input places, $Y \subseteq P$ is a set of output places, at that sets of input and output places do not intersect: $X \bigcap Y = \varnothing$, and, moreover, input places do not have input arcs and output places do not have output arcs: $\forall p \in X : {}^{\bullet}p = \varnothing$, $\forall p \in Y : p^{\bullet} = \varnothing$. Places of set $Q = P \setminus (X \bigcup Y)$ will be named an internal and places $C = X \bigcup Y$ – a contact.

Petri net $N' = (P', T', F')$ is a *subnet* of net N if $P' \subseteq P, T' \subseteq T, F' \subseteq F$.

Functional net $Z = (N', X, Y)$ will be named a *functional subnet* of net N and denoted as $Z \succ N$ if N' is subnet of N and, moreover, Z is connected with residuary part of net only by arcs incident to either input or output places, at that input places may have only input arcs and output places – only output arcs. Thus we have:

$$\forall p \in X : \{(p,t) \mid t \in T \setminus T'\} = \varnothing, \quad \forall p \in Y : \{(t,p) \mid t \in T \setminus T'\} = \varnothing,$$
$$\forall \in Q : \{(p,t) \mid t \in T \setminus T'\} = \varnothing \wedge \{(t,p) \mid t \in T \setminus T'\} = \varnothing .$$

Functional subnet $Z' \succ N$ is a *minimal* if it does not contain any other functional subnet of the source Petri net N.

Net generated by the indicated set of transitions $R \subseteq T$ will be denoted as $B(R)$.

The decomposition into functional subnets [7] has been investigated in [8]. Invariants of functional subnets were studied in [10,15]. Let us enumerate the most significant *properties of functional subnets*:

1) Functional subnet is generated by the set of its own transitions.
2) Set of minimal functional subnets $\Im = \{Z^j\}$, $Z^j \succ N$ defines the partition of set T into nonintersecting subsets T^j, such that $T = \bigcup_j T^j$, $T^j \bigcap T^k = \varnothing$, $j \neq k$.
3) Each functional subnet Z' of an arbitrary Petri net N is the sum (union) of finite number of minimal functional subnets. Union of subnets may be defined with the aid of operation of contact places fusion.
4) Each contact place of decomposed Petri net has no more than one input minimal functional subnet and no more than one output minimal functional subnet.
5) Petri net N is invariant iff all its minimal functional subnets Z^j, $Z^j \succ N$ are invariant and there is a common nonzero invariant of contact places.

4. FUNDAMENTAL EQUATIONS OF FUNCTIONAL SUBNETS

Let us consider the structure of system (1):

$$\bar{x} \cdot A = \Delta \bar{\mu}.$$

Each equation L_i: $\bar{x} \cdot A^i = \Delta \mu_i$, where A^i denotes i-th column of matrix A, corresponds to transition t_i (of dual net). Equation contains the terms for all the incident places. At that the coefficients are equals to weights of arcs and the terms for input places have sign minus and for output places – plus.

Therefore the system (1) may be represented as

$$L = L_1 \wedge L_2 \wedge ... \wedge L_n \tag{2}$$

Theorem 1. Solution $\bar{x}'$ of fundamental equation for Petri net N is the solution of fundamental equation for each of its functional subnets.

Proof. As $\bar{x}'$ is the solution of fundamental equation for Petri net N, so $\bar{x}'$ is a nonnegative integer solution of system (2) and consequently $\bar{x}'$ is a nonnegative integer solution for each of equations L_i. Thus $\bar{x}'$ is a solution for an arbitrary subset $\{L_i\}$.

According to property 1), a functional subnet Z', $Z' \succ N$ is generated by the set of its own transitions T'. Thus, an equation corresponding to a transition of subnet has the same form L_i as for the entire net, so subnet contains all the incident places of source net.

Therefore the system representing the fundamental equation for functional subnet Z', $Z' \succ N$ is a subset of set $\{L_i\}$ and vector $\bar{x}'$ is its solution. Consequently $\bar{x}'$ is the solution of fundamental equation for functional subnet Z'. Arbitrary choice of subnet $Z' \succ N$ in above reasoning proves the theorem. □

Theorem 2. Fundamental equation of Petri net is solvable if and only if it is solvable for each minimal functional subnet and a common solution for contact places exists.

Proof. We shall use equivalent transformations of systems of equations to not prove separately necessary and sufficient conditions. According to property 2), a set of minimal functional subnets $\Im = \{Z^j\}, Z^j \succ N$ of an arbitrary Petri net N defines a partition of set T into nonintersecting subsets T^j. Let number of minimal functional subnets equals k. As it was mentioned in the proof of theorem 1, equations contain the terms for all the incident places. Therefore,

$$L \quad \Leftrightarrow \quad L^1 \wedge L^2 \wedge \ldots \wedge L^k, \tag{3}$$

where L^j is a subsystem for a minimal functional subnet $Z^j, Z^j \succ N$. Notice that if L^j has not solutions, than L has not solutions also.

Let us a general solution for each functional subnet has the form

$$\bar{x}^j = \bar{x}'^j + \bar{u}^j \cdot G^j, \tag{4}$$

where $\bar{u}^j \cdot G^j$ is the general solution of homogeneous system, $\bar{x}'^j \in X'^j$, where X'^j is the set of minimal particular solutions of nonhomogeneous system of equations. According to (3):

$$L \quad \Leftrightarrow \quad \bar{x} = \bar{x}'^1 + \bar{u}^1 \cdot G^1 = \bar{x}'^2 + \bar{u}^2 \cdot G^2 = \ldots = \bar{x}'^k + \bar{u}^k \cdot G^k.$$

Therefore system

$$\bar{x} = \bar{x}'^1 + \bar{u}^1 \cdot G^1 = \bar{x}'^2 + \bar{u}^2 \cdot G^2 = \ldots = \bar{x}'^k + \bar{u}^k \cdot G^k \tag{5}$$

is equivalent to source system of equations (1). We shall demonstrate further that the solution of system (5) requires essentially smaller quantity of equations. Let us consider a set of places of Petri net N with the set of minimal functional subnets $\{Z^j \mid Z^j \succ N\}$:

$$P = Q^1 \cup Q^2 \cup \ldots \cup Q^k \cup C,$$

where Q^j is a set of internal places of subnet Z^j and C is a set of contact places. According to definition each internal place $p \in Q^j$ is incident only to transitions from set T^j. Thus x_p corresponding to this place is contained only in system L^j. Consequently, we have to solve only equations for contact places from set C.

Now we construct equations for contact places of net $p \in C$, so only they are incident more than one subnet. According to property 4), each contact place $p \in C$ is incident not more than two functional subnets. Therefore, we have equations

$$\bar{x}'^j_p + \bar{u}^j \cdot G^j_p = \bar{x}'^l_p + \bar{u}^l \cdot G^l_p, \tag{6}$$

where j, l is the numbers of minimal functional subnets incident to contact place $p \in C$ and G^j_p is a column of matrix G^j corresponding to place p. Equation (6) may be represented in form

$$\bar{u}^j \cdot G^j_p - \bar{u}^l \cdot G^l_p = \bar{x}'^l_p - \bar{x}'^j_p.$$

Thus, system

$$\begin{cases} x_p = \bar{x}'^j_p + \bar{u}^j \cdot G^j_p, \quad p \in Q^j \vee p \in C, \\ \bar{u}^j \cdot G^j_p - \bar{u}^l \cdot G^l_p = \bar{x}'^l_p - \bar{x}'^j_p, \quad p \in C \end{cases} \tag{7}$$

is equivalent to source system (1). This fact completes the proof of theorem. □

Notice that in both cases described in proof according to (7), we have to solve a linear homogeneous system of equations.

Corollary 1. To solve the fundamental equation of Petri net we may solve the fundamental equations of its minimal functional subnets and then to find a common solutions for contact places.

Corollary 2. Theorem 2 is valid also for an arbitrary set of functional subnets defining a partition of the set of transition of Petri net.

5. COMPOSITION OF FUNDAMENTAL EQUATIONS

Taking into consideration the results obtained in the previous section we may formulate a compositional method for solution of fundamental equation of Petri net:

- **Stage 0**. Construct a dual Petri net.
- **Stage 1**. Decompose dual Petri net into functional subnets.

- **Stage 2**. Calculate solutions for each of functional subnets – find general solutions of nonhomogeneous systems of equations (4).
- **Stage 3**. Compose subnets – find the common solution (6) for the set of contact places.

Note that stages 2, 3 consist in solution of systems of linear nonhomogeneous Diophantine equations in nonnegative integer numbers. For this purpose the methods described in [3,4,6,9] may be applied.

Let us extract out of system (7) equations for contact places

$$\bar{u}^{j} \cdot G_i^j - \bar{u}^{l} \cdot G_i^l = \bar{x}'^{l} - \bar{x}'^{j}.$$

Or in the matrix form

$$\left\| \bar{u}^{j} \quad \bar{u}^{l} \right\| \cdot \left\| \begin{matrix} G_i^j \\ -G_i^l \end{matrix} \right\| = \bar{b}_i', \; \bar{b}_i' = \bar{x}'^{l} - \bar{x}'^{j}$$

Let us enumerate all the variables $\bar{u}^{j}$ in such a way to obtain a united vector

$$\bar{u} = \left\| \bar{u}^{1} \quad \bar{u}^{2} \quad \ldots \quad \bar{u}^{k} \right\|$$

and to assemble the matrixes G_i^j, $-G_i^l$ in a united matrix K. Then we obtain system

$$\bar{u} \cdot K = \bar{b}'.$$

System obtained has the form (1), consequently, its general solution has the form (4):

$$\bar{u} = \bar{u}' + \bar{v} \cdot J. \tag{8}$$

Let us construct a united matrix G of solutions (4) of system (1) for all the functional subnets in such a manner that

$$\bar{x} = \bar{x}' + \bar{u} \cdot G. \tag{9}$$

We substitute (8) in (9):

$$\bar{x} = \bar{x}' + \left(\bar{u}' + \bar{v} \cdot J\right) \cdot G = \bar{x}' + \bar{u}' \cdot G + \bar{v} \cdot J \cdot G.$$

Thus

$$\bar{x} = \bar{x}'' + \bar{v} \cdot H, \; \bar{x}'' = \bar{x}' + \bar{u}' \cdot G, \; H = J \cdot G. \tag{10}$$

Since only equivalent transformations were involved, the reasoning represented above proves the following theorem.

Theorem 3. Expressions (10) represent a general solution of fundamental equation (1).

Now we estimate the total acceleration of calculations under the obtaining of invariants via decomposition. Let r be a maximal number either contact or internal places of subnets. Notice that $r \le n$. Then the complexity of fundamental equation solution for subnet may be estimated as $\sim 2^r$, since the complexity of decomposition according to [8] is polynomial.

Thus, the acceleration of computations is estimated as

$$2^n \big/ 2^r = 2^{n-r}. \tag{11}$$

Therefore, acceleration of computations obtained is exponential.

Notice that the exponential acceleration of computations represented with expression (11) is valid also in the case the general solutions for the functional subnets have more than one minimal particular solution. Really, let each of minimal functional subnets has not more than n minimal solutions. Then during calculation of common solutions for contact places we ought to solve n^2 systems and polynomial multiplier may be omitted in the comparison estimations of exponential functions.

6. AN EXAMPLE OF FUNDAMENTAL EQUATION SOLUTION

Let us check the reachability of marking $\bar{\mu} = (0,2,1,0,4)$ in Petri net N_1 (Fig. 1). Thus $\Delta\bar{\mu} = (-1,2,1,-1,4)$.

Stages 0,1. Dual Petri net $\tilde{N}_1$ (Fig. 2) is decomposed into four minimal functional subnets Z^1, Z^2, Z^3, Z^4 completely defined by the subsets of its transitions: $T^1 = \{t_1\}$, $T^2 = \{t_2, t_3\}$, $T^3 = \{t_5\}$, $T^4 = \{t_4\}$.

Stage 2.

Z^1: $\quad \left\{ -x_1 + x_4 + x_6 = -1; \right. \qquad \bar{x} = (1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0) + (u_1^1, u_2^1) \cdot \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$

Z^2: $\quad \begin{cases} 3x_1 - 3x_2 - x_5 = 2, \\ x_1 - x_2 - x_3 = 1; \end{cases} \qquad \bar{x} = (1 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0) + (u_1^2, u_2^2) \cdot \begin{pmatrix} 1 & 0 & 1 & 0 & 3 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$

Z^3: $\quad \left\{ 6x_2 + x_5 - 6x_6 = 4; \right. \qquad \bar{x} = (0 \quad 0 \quad 0 \quad 0 \quad 4 \quad 0) + (u_1^3, u_2^3) \cdot \begin{pmatrix} 0 & 0 & 0 & 0 & 6 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}.$

Z^4: $\quad \left\{ x_3 - 2x_4 = -1; \right. \qquad \bar{x} = (0 \quad 0 \quad 1 \quad 1 \quad 0 \quad 0) + (u_1^4) \cdot \begin{pmatrix} 0 & 0 & 2 & 1 & 0 & 0 \end{pmatrix}.$

Stage 3.

$$\begin{cases} u_1^1 + u_2^1 - u_1^2 - u_2^2 = 0, \\ u_2^2 - u_2^3 = 0, \\ u_1^2 - 2u_1^4 = 1, \\ u_1^1 - u_1^4 = 1, \\ 3u_1^2 - 6u_1^3 = 3, \\ u_2^1 - u_1^3 - u_2^3 = 0; \end{cases} \qquad \begin{aligned} \bar{u} &= (u_1^1 \quad u_2^1 \quad u_1^2 \quad u_2^2 \quad u_1^3 \quad u_2^3 \quad u_1^4) = \\ &= (1 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0) + (v_1, v_2) \cdot \begin{pmatrix} 1 & 1 & 2 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}. \end{aligned}$$

$$\bar{x} = (2 \quad 0 \quad 1 \quad 1 \quad 4 \quad 0) + (v_1, v_2) \cdot \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ 2 & 0 & 2 & 1 & 6 & 1 \end{pmatrix}.$$

Notice that the general solution of homogeneous equation constitutes *t*-invariant of Petri net. On the minimal solution $\bar{x}' = (2,0,1,1,4,0)$ we may construct the friable sequence $\sigma = t_1 t_5 t_5 t_5 t_3 t_4 t_1 t_5$. Therefore, marking $\bar{\mu} = (0,2,1,0,4)$ is reachable in net N_1.

In this tiny example all the places are contact, so we have not obtained an acceleration of computations. For real-life examples the accelerations may become rather considerable [11,12,16].

7. CONCLUSION

The complexity of Petri net fundamental equation solution is exponential in general case. This fact makes the analysis of real-life objects difficult. The technique proposed and studied in present paper allows the acceleration of the solution of fundamental equation. This technique is based on the decomposition of Petri net into functional subnets. The acceleration obtained is exponential with respect to the number of nodes of source Petri net.

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