

Vistula, IT Faculty, 2014

# Algorithms and Complexity

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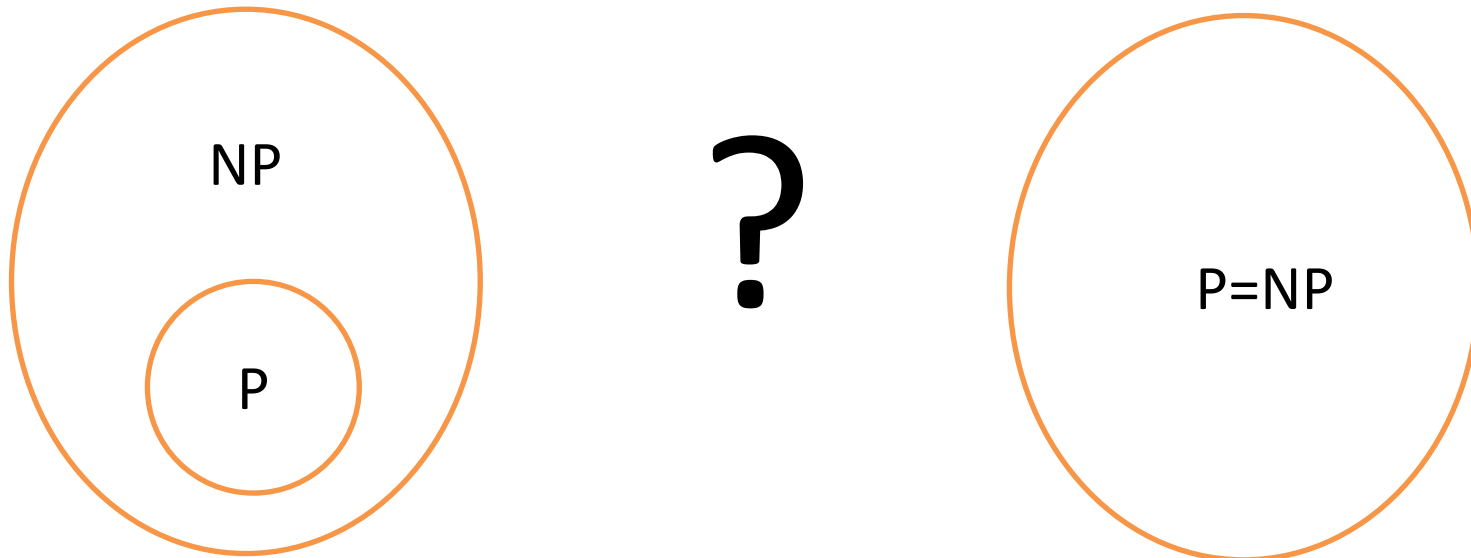
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## Lecture 12:

## NP-complete problems

## NP=P ?

- For NP only exponential algorithms are known
- Noone has proven that there is at least one problem in NP that does not belong to P



## NP-completeness

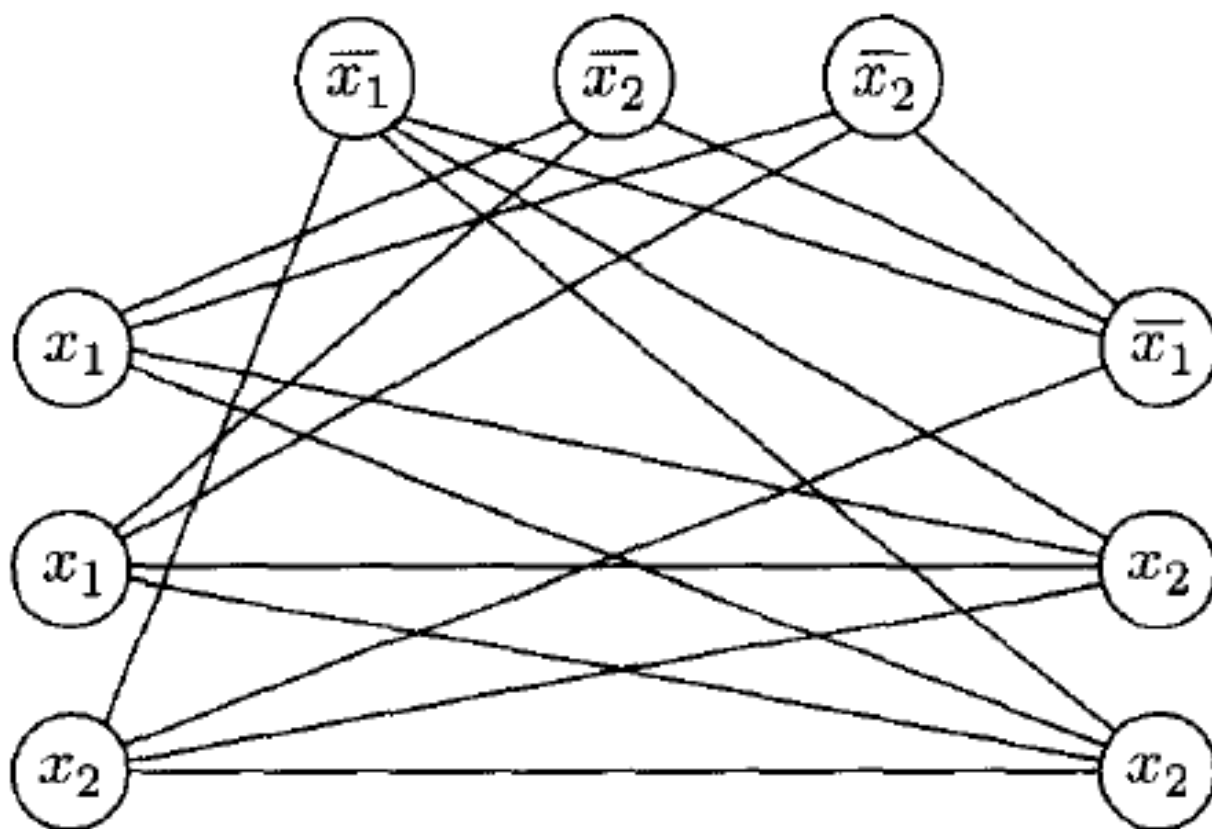
- A NP-problem Z is NP-complete if any other NP-problem Z' is polynomially reducible to Z
- Polynomial reduction of MAP – polynomial-time TM
- An example of reduction: 3SAT  $\rightarrow$  k-clique
- 3-CNF-Formula:

$$f = (a_1 \vee b_1 \vee c_1) \wedge (a_2 \vee b_2 \vee c_2) \dots \wedge (a_k \vee b_k \vee c_k)$$
$$a_i, b_i, c_i \in \{x_j, \bar{x}_j\}$$

Graph contains  $3k$  vertices labeled with the content of terms  
Edges connect all vertices except of: (a) same term, (b) same letter.

## 3SAT -> k-clique

$$\phi = (x_1 \vee x_1 \vee x_2) \wedge (\overline{x_1} \vee \overline{x_2} \vee \overline{x_2}) \wedge (\overline{x_1} \vee x_2 \vee x_2).$$



## Cook theorem

The problem of determining whether a Boolean expression is satisfiable is NP-complete

1. SAT in NP
2. Any NP is reducible to SAT

Let  $Z$  an arbitrary NP problem. There is NDTM that solves it.

Let for a given word  $w$  NDTM decides it in time  $k$ .

Represent the sequence of configuration as  $k \times k$  matrix of cells

$c_{i,j}$

Logical variables  $x_{i,j,s} = 1$  iff  $c_{i,j}$  contains symbol  $s$

## Boolean formula on NDTM run

$$\varphi = \varphi_{\text{cell}} \wedge \varphi_{\text{start}} \wedge \varphi_{\text{move}} \wedge \varphi_{\text{accept}}$$

$$\phi_{\text{cell}} = \bigwedge_{1 \leq i, j \leq n^k} \left[ \left( \bigvee_{s \in C} x_{i,j,s} \right) \wedge \left( \bigwedge_{\substack{s, t \in C \\ s \neq t}} (\overline{x_{i,j,s}} \vee \overline{x_{i,j,t}}) \right) \right].$$

$$\begin{aligned} \phi_{\text{start}} = & x_{1,1,\#} \wedge x_{1,2,q_0} \wedge \\ & x_{1,3,w_1} \wedge x_{1,4,w_2} \wedge \dots \wedge x_{1,n+2,w_n} \wedge \\ & x_{1,n+3,\sqcup} \wedge \dots \wedge x_{1,n^k-1,\sqcup} \wedge x_{1,n^k,\#} . \end{aligned}$$

$$\phi_{\text{accept}} = \bigvee_{1 \leq i, j \leq n^k} x_{i,j,q_{\text{accept}}} .$$

$$\phi_{\text{move}} = \bigwedge_{1 < i \leq n^k, 1 < j < n^k} (\text{the } (i, j) \text{ window is legal})$$

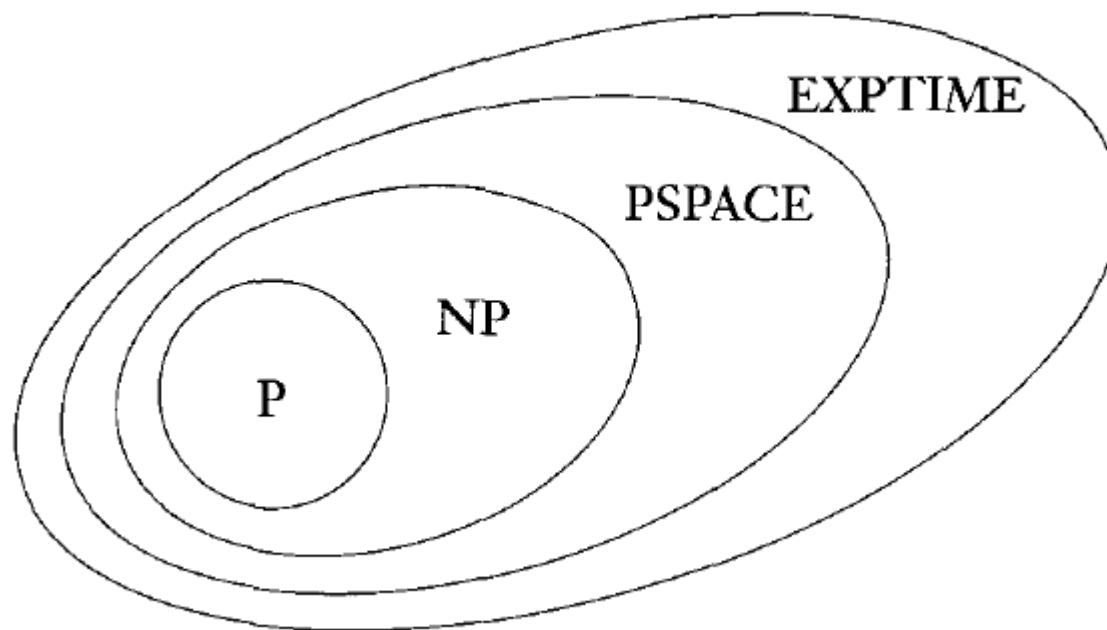
## List of NP-complete problem

- Satisfiability of Boolean formula
- Three satisfiability
- Clique
- Vertex cover
- Hamilton path
- Subset sum
- k-coloring
- ...

## Space complexity classes

- $\text{SPACE}(f(x))$  – decidable by TM
- $\text{NSPACE}(f(x))$  – decidable by NDTM

**Savitch's theorem** For any<sup>1</sup> function  $f: \mathcal{N} \longrightarrow \mathcal{R}^+$ , where  $f(n) \geq n$ ,  
$$\text{NSPACE}(f(n)) \subseteq \text{SPACE}(f^2(n)).$$





## How to do with NP-complete problems

- Divide and conquer
- Branch and bound
- Dynamic programming
- Local search
- Heuristic solutions
- Randomization